# Evacuation and a Geometric Construction for Fibonacci Tableaux

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**Abstract**

Tableaux have long been used to study combinatorial properties of permutations and multiset permutations. Discovered independently by Robinson and Schensted and generalized by Knuth, the Robinson-Schensted correspondence has provided a fundamental tool for relating permutations to tableaux. In 1963, Schützenberger defined a process called evacuation on standard tableaux which gives a relationship between the pairs of tableaux $(P, Q)$ resulting from the Schensted correspondence for a permutation and both the reverse and the complement of that permutation. Viennot gave a geometric construction for the Schensted correspondence and Fomin described a generalization of the correspondence which provides a bijection between permutations and pairs of chains in Young's lattice.

In 1975, Stanley defined a Fibonacci lattice and in 1988 he introduced the idea of a differential poset. Roby gave an insertion algorithm, analogous to the Schensted correspondence, for mapping a permutation to a pair of Fibonacci tableaux. The main results of this paper are to give an evacuation algorithm for the Fibonacci tableaux that is analogous to the evacuation algorithm on Young tableaux and to describe a geometric construction for the Fibonacci tableaux that is similar to Viennot's geometric construction for Young tableaux.

## 1 Introduction

The Robinson-Schensted correspondence ([3], [5], [8]) gives a bijection between permutations and pairs of standard tableaux $(P, Q)$ of the same shape $\lambda$. Schützenberger [9] defined a process called evacuation on standard tableaux which gives a relation between the pairs of tableaux $(P,\ Q)$ for a permutation $\pi$, the reverse permutation $\pi^r$ and the complement permutation $\pi^c$. Viennot [13] gave a geometric construction for the Schensted correspondence which gives a relation between the $P$ and $Q$ tableaux of $\pi$ and its inverse permutation $\pi^{-1}$. Fomin ([1], [2]) described a generalization of the Schensted correspondence which provides a bijection between a permutation $\pi$ and pairs of chains in Young's lattice.

In 1975, Stanley [10] introduced the idea of a Fibonacci lattice which he called $Fib(1)$ and in 1988 [11] he introduced the idea of differential posets, which both generalized his Fibonacci lattice $Fib(1)$ to $Fib(r)$ and gave a second Fibonacci lattice called $Z(r)$. In this paper, we will deal only with the second Fibonacci lattice $Z(r)$ and only the case $r = 1$. Fomin's generalization of the Schensted correspondence gives a bijection between any permutation $\pi$ and pairs of saturated chains in $Z(r)$. These pairs of chains can be translated into pairs of tableaux $(\hat{P}, \hat{Q})$ of Fibonacci shape called Fibonacci path tableaux. Roby [6] gave an insertion algorithm (analogous to the Schensted correspondence) for mapping a permutation $\pi$ to a pair of Fibonacci tableaux $(P, Q)$ where $P$ is called a Fibonacci insertion tableau ($\hat{P} \neq P$)and $Q$ is the same as the $\hat{Q}$ path tableau. The main result of this paper, given in Section 5, is to define an evacuation algorithm which relates the Fibonacci path tableau $\hat{P}$ to the Fibonacci insertion tableau $P$ for any permutation $\pi$. In addition, a geometric construction for the insertion algorithm (analogous to Viennot's geometric construction) is described in Section 4. These results are given for the Fibonacci lattice $Z(1)$ and generalize easily to $Z(r)$. The necessary background is given in Sections 2 and 3.

## 2 The Fibonacci Lattice

In this section, we give the basic definitions needed for this paper. The interested reader is encouraged to read Chapter 5 of Bruce Sagan's *The Symmetric Group, 2nd Edition* [7] for general reference.

We define a *differential poset* as follows:

**Definition 1.** *A differential poset is a poset which satisfies the following three conditions:*

1. $P$ has a $\hat{0}$ element, is graded and is locally finite.

2. If $x \neq y$ and there are exactly $k$ elements in $P$ which are covered by $x$ and by $y$, then there are exactly $k$ elements in $P$ which cover both $x$ and $y$.

3. For $x \in P$, if $x$ covers exactly $k$ elements of $P$, then $x$ is covered by exactly $k + 1$ elements of $P$.

The classic example of a differential poset is Young's lattice, which is the poset of the set of partitions together with the binary relation $\lambda \leq \mu$ if and only if $\lambda_i \leq \mu_i$ for all $i$.

A second kind of differential poset is the Fibonacci differential poset. The general definition of a Fibonacci $r$-differential poset was given by Richard Stanley in [11] (Definition 5.2). Here we give only the definition for $r = 1$.

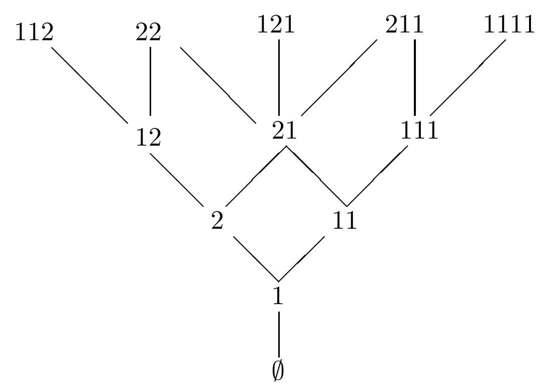
Let $A = \{1, 2\}$ and let $A^*$ be the set of all finite words $a_1 a_2 \cdots a_k$ of elements of $A$ (including the empty word).

**Definition 2.** *The Fibonacci differential poset $Z(1)$ has as its set of elements the set of words in $A^*$. If $w \in Z(1)$, then define $z$ to be covered by $w$ (i.e. $z \lessdot w$) in $Z(1)$ if either:*

1. $z$ is obtained from $w$ by changing a 2 to a 1 if the only letters to the left of this 2 are also 2's, or

2. $z$ is obtained from $w$ by deleting the leftmost 1.

If we let $F_i$ be the set of elements of $Z(1)$ which have rank $i$, then we can determine $|F_i|$ recursively. To obtain the elements of $F_i$, we prepend a 2 to those elements of $F_{i-2}$ and prepend a 1 to the elements of $F_{i-1}$. Thus $|F_i| = |F_{i-2}| + |F_{i-1}|$, hence the name Fibonacci differential poset. The first four rows of the Fibonacci lattice $Z(1)$ are shown below.




## 3 Fibonacci Tableaux and Chains in the Fibonacci Lattice

Given a permutation $\pi \in S_n$, we can obtain a pair of standard Young tableaux $(P, Q)$, both of the same shape $\lambda$, through a process called the Robinson-Schensted correspondence. In addition, we can obtain a pair of chains in Young's lattice through a method of Fomin's and we can map this pair of chains directly to the pair of Young tableaux resulting from the Robinson-Schensted correspondence.

In a manner analogous to that for Young tableaux, we can use Fomin's generalization to relate a pair of chains in the Fibonacci lattice to a pair of Fibonacci tableaux $(\hat{P}, \hat{Q})$ of the same shape, called Fibonacci path tableaux. In addition, Roby [6] gave an insertion algorithm similar to the Robinson-Schensted correspondence which provided a bijection between permutations and pairs of Fibonacci tableaux $(P, Q)$ of the same shape for which the $P$ tableau is called a Fibonacci insertion tableau.

Given a permutation $\pi \in S_n$, we can find a pair of chains in the Fibonacci lattice by first representing the permutation by a square diagram. Represent $\pi = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array}$ in a square diagram by placing an $X$ in column $i$ and row $x_i$ (indexed from left to right, bottom to top) if $\begin{array}{c} i \\ x_i \end{array}$ is a column in the permutation $\pi$. For example, for the permutation

$$\pi = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 1 & 5 & 6 & 4 & 3 \end{array}$$

we obtain the following square diagram:

| | | | | | | |
|---|---|---|---|---|---|---|
| | X | | | | | |
| | | | | X | | |
| | | | X | | | |
| | | | | | X | |
| | | | | | | X |
| X | | | | | | |
| | | X | | | | |

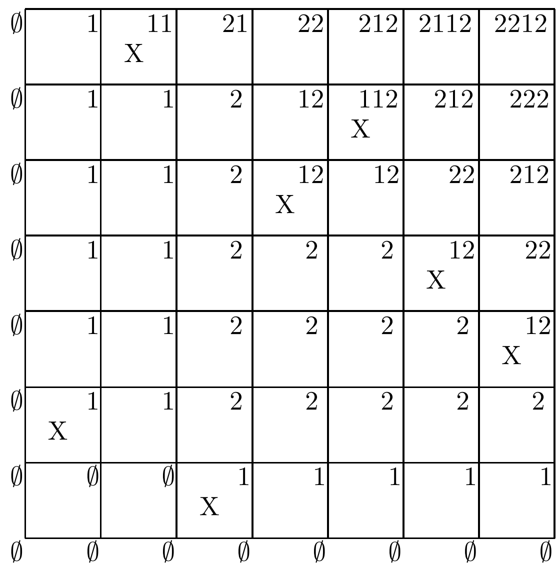
Fomin [2] described the following method for using the square diagram to create a pair of saturated chains in the Fibonacci lattice. Begin by placing $\emptyset$'s along the lower edge and the left edge at each corner. In order to label the remaining corners in the diagram, we will define several rules which we will call a *growth function* or *growth diagram*. If we have

| | |
|---|---|
| $\mu_1$ | $\lambda$ |
| $\nu$ | $\mu_2$ |

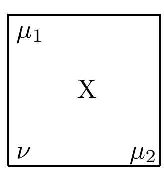
with each side of the square representing a cover relation in the Fibonacci lattice or an equality, then:

1. If $\mu_1 \gtrdot \nu$ and $\mu_2 = \nu$ then $\lambda = \mu_1$ (and similarly for $\mu_1$ and $\mu_2$ interchanged).
2. If $\mu_1 \gtrdot \nu$, $\mu_2 \gtrdot \nu$ then $\lambda$ is obtained from $\nu$ by prepending a 2.
3. If $\mu_1 = \nu = \mu_2$ and the box does contain an $X$, then obtain $\lambda$ from $\nu$ by prepending a 1.
4. If $\mu_1 = \nu = \mu_2$ and the box does not contain an $X$, then $\lambda = \nu$.

By following this procedure on our previous example, we obtain the complete growth diagram:




where the uppermost edge and the rightmost edge each represent a chain in the Fibonacci lattice from $\emptyset$ to 2212. From the definition of the growth function, one can see that at any position $\nu$ in the growth diagram, the size and shape of $\nu$ is completely determined by the number and relative position of the X's below and to the left of the position of $\nu$. For any square containing an X,




there are no X's immediately below or immediately to the left of the X in this square since there is only one X per row and one X per column in the square diagram. Since the number and relative position of the X's below and to the left of $\nu$, $\mu_1$, and $\mu_2$ is the same, then $\nu = \mu_1 = \mu_2$ around any square containing an X.

Fomin proved in [2] that this growth function produces a saturated chain in the Fibonacci differential poset along both the right edge and the top edge of the diagram. We can translate a saturated chain $\nu = (\emptyset, \nu_1, \nu_2, \ldots, \nu_k = \nu)$ in the Fibonacci lattice into a Fibonacci tableau by placing an $i$ in $\nu_i/\nu_{i-1}$, i.e. in the new square that was added to $\nu_{i-1}$ to form $\nu_i$. For the example above, we obtain the tableau defined as $\hat{P}$ from the right edge and the tableau defined as $\hat{Q}$ from the top edge:

$$\hat{P} \quad = \quad \begin{array}{cccc} 4 & 6 & & 2 \\ 3 & 5 & 7 & 1 \end{array}, \qquad \hat{Q} \quad = \quad \begin{array}{cccc} 3 & 7 & & 4 \\ 2 & 6 & 5 & 1 \end{array}$$

Roby [6] gave a description of an insertion algorithm that gives a bijection between permutations in $S_n$ and pairs of Fibonacci tableaux $(P, Q)$ for which the $P$ tableau is called a Fibonacci insertion tableau and the $Q$ tableaux is called a Fibonacci recording tableau. To apply Roby's insertion algorithm to a permutation, we will construct a sequence $\{(P_i, Q_i)\}_{i=0}^{n}$, where $(P_0, Q_0) = (\emptyset, \emptyset)$ and $(P_i, Q_i)$ are the tableaux resulting from the insertion of $x_i$ into $P_{i-1}$, in the following manner.

1. Compare $x_i$ to the value $t_1$ of the number in the leftmost square in the bottom row of $P_{i-1}$.

2. If $x_i > t_1$ then add a square to the left of the bottom row and put the value $x_i$ inside. This tableau is the new tableau $P_i$, and to form $Q_i$, a tableau of the same shape as $P_i$, place an $i$ in this newly created square.

3. If $x_i < t_1$, then place $x_i$ in the square directly above $t_1$. If the square above $t_1$ was previously empty, then this new tableau is $P_i$ and we obtain $Q_i$ by placing an $i$ in this corresponding new box. If the square was not previously empty, but contained an element $b$, then $b$ is bumped out of the square.

4. Continue by inductively inserting $b$ into the tableau to the right of the first column, comparing $b$ to the element $t_2$ in the box to the right of $t_1$ in the bottom row and repeating steps 1-3 with $b$ and $t_2$.

For example, using the same permutation as previously,

$$\pi \quad = \quad \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 1 & 5 & 6 & 4 & 3 \end{array}$$

we obtain the sequence:

$$P_i \quad = \quad 2, \quad \begin{array}{cc} 7 & 2 \end{array}, \quad \begin{array}{cc} 1 & \\ 7 & 2 \end{array}, \quad \begin{array}{cc} 5 & 1 \\ 7 & 2 \end{array}, \quad \begin{array}{ccc} 6 & & 1 \\ 7 & 5 & 2 \end{array}, \quad \begin{array}{cccc} 4 & & & 1 \\ 7 & 6 & 5 & 2 \end{array}, \quad \begin{array}{cccc} 3 & 4 & & 1 \\ 7 & 6 & 5 & 2 \end{array}$$

$$Q_i \quad = \quad 1, \quad \begin{array}{cc} 2 & 1 \end{array}, \quad \begin{array}{cc} 3 & \\ 2 & 1 \end{array}, \quad \begin{array}{cc} 3 & 4 \\ 2 & 1 \end{array}, \quad \begin{array}{ccc} 3 & & 4 \\ 2 & 5 & 1 \end{array}, \quad \begin{array}{cccc} 3 & & & 4 \\ 2 & 6 & 5 & 1 \end{array}, \quad \begin{array}{cccc} 3 & 7 & & 4 \\ 2 & 6 & 5 & 1 \end{array}$$

thus

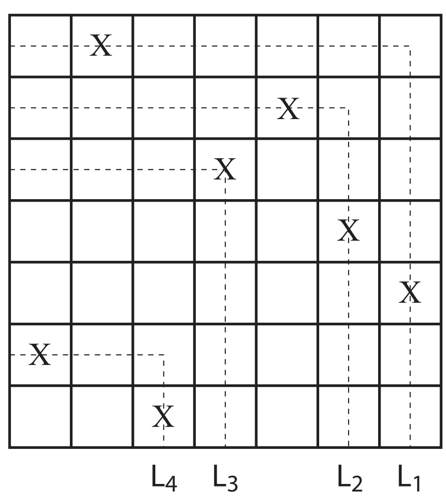
$$P = \begin{array}{cccc} 3 & 4 & & 1 \\ 7 & 6 & 5 & 2 \end{array}, \qquad Q = \begin{array}{cccc} 3 & 7 & & 4 \\ 2 & 6 & 5 & 1 \end{array}$$

Roby [6] proved that the $Q$ tableau is exactly the same as the $\hat{Q}$ path tableau for any permutation. However, unlinke the case for Young tableaux, the $P$ tableau is not equal to the $\hat{P}$ tableau. The evacuation procedure defined in Section 5 will give a relation between these tableaux.

# 4 A Geometric Construction

We will now give a new geometric construction, similar to Viennot's geometric construction for Young tableaux, through which we can obtain the insertion tableau $P$ directly from the square diagram (analogous to Viennot's "shadow lines").

To draw the lines into the square diagram, begin at the top row and draw a broken line $L_1$ through the X in the top row and the X in the rightmost column. The second broken line $L_2$ will be drawn through the row containing the highest X not already on a line and the rightmost column containing an X not already on a line. Continue in this manner until there are no more X's available. For example, for the permutation $\pi = 2715643$, the lines look like:

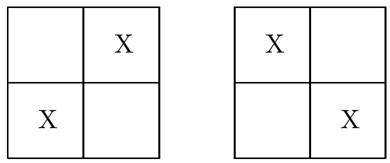



**Theorem 1.** *The row numbers of the X's on $L_i$ give the elements in the ith column of the insertion $P$ tableau, with the larger row number (if there are two X's on the line) in the bottom row and the smaller row number in the top row.*

For the example above, the shadow lines give the $P$ tableau

$$P(\pi) = \begin{array}{cccc} 3 & 4 & & 1 \\ 7 & 6 & 5 & 2 \end{array}$$

*Proof.* We will prove this result by induction on the size of $\pi$. If $\pi \in S_1$ then $\pi = 1$ and using the shadow lines on the square diagram for $\pi$ gives the $P$ tableau 1, which is equal to the insertion tableau $P(\pi)$. If $\pi \in S_2$ then either $\pi = 12$ or $\pi = 21$, which are represented by the square diagrams:



$\pi = 12$ $\qquad$ $\pi = 21$

If $\pi = 12$, then the insertion tableau $P(12) = \begin{array}{cc} 2 & 1 \end{array}$ and one can easily check that this is the same as the tableau determined by the shadow lines. If $\pi = 21$, then the insertion tableau $P(21) = \begin{array}{c} 1 \\ 2 \end{array}$ and again one can easily check that this is the tableau determined by the shadow lines.

Now assume that the tableau determined by the shadow lines for $\sigma \in S_k$ with $k < n$ is equal to the insertion tableau $P(\sigma)$. Let $\pi \in S_n$ and represent the permutation $\pi$ with a square diagram. Draw $L_1$. If there is an X in the upper right corner of the square diagram, then $L_1$ only passes through one X.

Since an X in the upper right corner implies that $n$ is the last number in the permutation $\pi$, we can write $\pi = \pi_{n-1}n$, where $\pi_{n-1}$ represents the first $n-1$ digits in the permutation $\pi$. Since $n$ is the last number in the permutation, when we apply the insertion algorithm to $\pi$, $n$ is the last number inserted into the tableau. Thus the insertion tableau $P$ equals $nP_{n-1}$ where $P_{n-1}$ is the insertion tableaux for $\pi_{n-1}$. Thus the fact that the line $L_1$ drawn in the $n$th row and $n$th column only passes through one X corresponds to the fact that the first column of the $P$ tableau has only one element and that element is $n$.

If there is no X in the upper right square, then $L_1$ passes through two X's, one in row $n$ and one in column $n$ and row $a$ (counting from the bottom) with $a < n$. Since this means that $a$ is the last element in the permutation $\pi$, then $a$ is the last element inserted into the $P$ tableau. Due to the method of insertion, the element $n$, which corresponds to the X in the uppermost row, is always in the lower left position of $P$. Thus when $a$ is inserted into the tableau, it is inserted into the second row above the $n$, possibly bumping an element $b$ to the second column. The resulting $P$ tableau has $\begin{array}{c} a \\ n \end{array}$ in the first column, corresponding to the fact that $L_1$ passes through two X's, one in row $n$ and one in row $a$.

It remains to show that the rest of the $P$ tableau can be determined by removing the $n$th row and the $n$th column from the square diagram, since these elements are in the first column of $P$, and applying the inductive hypothesis to the remaining diagram. Let the permutation $\pi$ be written as

$$\pi = \begin{array}{cccccccccc} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n-1 & n \\ x_1 & x_2 & \cdots & x_{i-1} & n & x_{i+1} & \cdots & x_{n-1} & a \end{array}.$$

Recall that $P_i$ is the insertion tableau of the first $i$ elements $x_1 x_2 \cdots x_{i-1} n$. By definition of the insertion algorithm, $P_i = nP_{i-1}$ and since $x_k < n$ $\forall k \neq i$

then $x_{i+1} < n$ so $P_{i+1} = \begin{matrix} x_{i+1} \\ n \end{matrix} \; P_{i-1}$.

When $x_{i+2}$ is inserted into $P_{i+1}$, $x_{i+1}$ is bumped out of column one and inserted into the tableau to the right, which is $P_{i-1}$. When $x_{i+3}$ is inserted, $x_{i+2}$ is bumped out of column one and inserted into the tableau to the right. At the last step, $a$ bumps $x_{n-1}$ from the first column and $x_{n-1}$ is then inserted into the tableau to the right. The resulting tableau is thus the same as the tableau obtained by placing the column $\begin{matrix} a \\ n \end{matrix}$ in front of the tableau obtained from the insertion of

$$\sigma = \begin{matrix} 1 & 2 & \cdots & i-1 & i & \cdots & n-2 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n-1} \end{matrix},$$

which is obtained from $\pi$ by removing $n$ and $a$. The growth diagram for $\sigma$ is the same as the growth diagram for $\pi$ with the top row and rightmost column removed and any empty rows and columns removed (since empty rows and empty columns do not affect the growth diagram). Inductively, we can now apply the above conditions to this new growth diagram to determine the elements in the second column of $P$ and continue to determine the complete insertion tableau $P(\pi)$. □

# 5 Evacuation

Notice that the $P$ and the $Q$ Fibonacci tableaux obtained through the insertion algorithm have the same shape and the $\hat{P}$ and the $\hat{Q}$ Fibonacci tableaux obtained as path tableaux from the growth diagram have the same shape. Since Roby [6] proved that $Q = \hat{Q}$ then we know that the shape of $P$ and $\hat{P}$ is the same. The tableaux $P$ and $\hat{P}$ are related through a process of evacuation similar to the evacuation process for Young tableaux.

Compute the evacuation of an insertion tableau $P$ in the following manner.

1. Erase the number in the leftmost position in the bottom row. This will necessarily be the largest number in $P$.

2. Compare the numbers immediately above and immediately to the right of this empty box. Slide the larger of these two numbers into the leftmost position in the bottom row, leaving a new empty box behind.

3. For this newly empty box, compare the numbers immediately above and to the right and slide the larger of the two numbers into the empty box, leaving another empty box behind.

4. If an empty box has a number immediately above it but no number immediately to the right of it, slide the number above it into the empty box, creating a newly empty box in the top row of the column.

5. Continue in this manner until reaching a box that has no number immediately above it. At this point, remove the empty box from the tableau

and if this results in an empty column in the middle of the tableau, slide all remaining columns one column to the left so that the result has the shape of a Fibonacci tableau. Call this remaining tableau $P^{(1)}$.

6. In a new tableau of the same shape as $P$, denoted by $\tilde{P}$, put an $n$ in the position of the last empty box.
7. Create $P^{(2)}$ by repeating the above procedure on $P^{(1)}$. At step 6, label the position of the last empty box with an $n-1$ in the tableau $\tilde{P}$. Continue until $P^{(n)} = \emptyset$ and $\tilde{P}$ is a Fibonacci tableau containing the numbers 1 through $n$. The final tableau $\tilde{P}$ is called the evacuation tableau $ev(P)$.

For example, using

$$P(\pi) \quad = \quad \begin{array}{cccc} 3 & 4 & & 1 \\ 7 & 6 & 5 & 2 \end{array}$$

the first sequence of steps is

$$\begin{array}{cccc} 3 & 4 & & 1 \\ \bullet & 6 & 5 & 2 \end{array} \qquad \begin{array}{cccc} 3 & 4 & & 1 \\ 6 & \bullet & 5 & 2 \end{array} \qquad \begin{array}{cccc} 3 & 4 & & 1 \\ 6 & 5 & \bullet & 2 \end{array}$$

and thus after one step of the evacuation procedure, $\tilde{P}$ looks like

$$\begin{array}{cccc} \bullet & \bullet & & \bullet \\ \bullet & \bullet & 7 & \bullet \end{array}$$

All of the steps in the evacuation of $P$ and the development of $\tilde{P}$ are shown in the following example:

$$P^{(k)}: \quad \begin{array}{cccc} 3 & 4 & & 1 \\ 7 & 6 & 5 & 2 \end{array} \quad \begin{array}{ccc} 3 & 4 & 1 \\ 6 & 5 & 2 \end{array} \quad \begin{array}{ccc} 3 & & 1 \\ 5 & 4 & 2 \end{array} \quad \begin{array}{cc} 3 & 1 \\ 4 & 2 \end{array}$$

$$\begin{array}{cc} & 1 \\ 3 & 2 \end{array} \qquad \begin{array}{c} 1 \\ 2 \end{array} \qquad \begin{array}{c} \\ 1 \end{array}$$

$$\tilde{P}: \quad \begin{array}{cccc} \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array} \quad \begin{array}{cccc} \bullet & \bullet & & \bullet \\ \bullet & \bullet & 7 & \bullet \end{array} \quad \begin{array}{cccc} \bullet & 6 & & \bullet \\ \bullet & \bullet & 7 & \bullet \end{array} \quad \begin{array}{cccc} \bullet & 6 & & \bullet \\ \bullet & 5 & 7 & \bullet \end{array}$$

$$\begin{array}{cccc} 4 & 6 & & \bullet \\ \bullet & 5 & 7 & \bullet \end{array} \quad \begin{array}{cccc} 4 & 6 & & \bullet \\ 3 & 5 & 7 & \bullet \end{array} \quad \begin{array}{cccc} 4 & 6 & & 2 \\ 3 & 5 & 7 & \bullet \end{array}$$

Completing the last slide, we have

$$ev(P) \quad = \quad \begin{array}{cccc} 4 & 6 & & 2 \\ 3 & 5 & 7 & 1 \end{array}$$

**Theorem 2.** *Let $(\hat{P}(\pi), \hat{Q}(\pi))$ be the pair of Fibonacci path tableaux obtained from the permutation $\pi \in S_n$ by using the growth diagram and let $(P(\pi), Q(\pi))$*

*be the Fibonacci insertion tableaux obtained from the permutation $\pi$ using Roby's insertion method. Then*

$$ev(P(\pi)) = \hat{P}(\pi)$$

*Proof.* We will prove that $ev(P(\pi)) = \hat{P}(\pi)$ by induction. If the length of $\pi$ is 1, then the path tableau $\hat{P}$ is 1 and the insertion tableau $P$ is 1, so $ev(P) = 1$. Thus $\hat{P}(1) = ev(P(1))$.

Assume that for $\sigma \in S_k$ with $k < n$, $ev(P(\sigma)) = \hat{P}(\sigma)$ and let the length of $\pi$ be $n$.

Case 1: Suppose the square in the uppermost, rightmost corner of the diagram for $\pi$ contains an X.

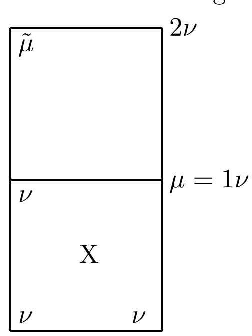
As previously stated, an X in this square implies that $n$ is the last number in the permutation $\pi$, so $\pi = \pi_{n-1}n$ where $\pi_{n-1}$ represents the first $n-1$ digits in the permutation $\pi$. From the square diagram, we have that $\hat{P} = n\hat{P}_{n-1}$ where $\hat{P}_{n-1}$ is the path tableau of shape $\nu$ obtained from $\pi_{n-1}$. Since $n$ is the last number in the permutation $\pi$, when we apply the insertion algorithm, $n$ is the last number inserted into the tableau. Thus the insertion tableau $P = nP_{n-1}$ where $P_{n-1}$ is the insertion tableaux for $\pi_{n-1}$. Following the evacuation procedure, the $n$ is simply removed from $P$ and $ev(P) = n\ ev(P_{n-1})$. Since $\pi = \pi_{n-1}n$, we know $\pi_{n-1} \in S_{n-1}$ and $\hat{P}_{n-1}$ is the path tableau obtained from $\pi_{n-1}$, so we can use our inductive hypothesis that $\hat{P}_{n-1} = ev(P_{n-1})$. Thus

$$ev(P) = n\ ev(P_{n-1}) = n\hat{P}_{n-1} = \hat{P}.$$

Case 2: Suppose the X in the $n$th column of the square diagram is in row $n-1$. In this case, the permutation $\pi$ looks like:

$$\pi = \begin{array}{cccccccc} 1 & 2 & \cdots & i & i+1 & \cdots & n-1 & n \\ x_1 & x_2 & \cdots & n & x_{i+1} & \cdots & x_{n-1} & n-1 \end{array}$$

and the top two squares in the last column of the growth diagram look like:



Here $\mu = 1\nu$ and $\lambda = 2\nu$ differ by the square in the first column, top row and $\mu$ and $\nu$ differ by the square in the first column, bottom row. Thus $\hat{P} = \begin{array}{c} n \\ n-1 \end{array} \hat{P}_{n-2}$, where $\hat{P}_{n-2}$ is the path tableau of shape $\nu$ obtained from the first $n-2$ rows of the growth diagram. The first $n-2$ rows have columns $i$ and $n$ empty and are the growth diagram for

$$\sigma = \begin{array}{ccccccc} 1 & 2 & \cdots & i-1 & i+1 & \cdots & n-1 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n-1} \end{array}$$

and $\hat{P}_{n-2}$ is then the path tableau of shape $\nu$ for $\sigma$. Removing empty columns from the first $n-2$ rows gives the permutation

$$\hat{\sigma} = \begin{array}{ccccccc} 1 & 2 & \cdots & i-1 & i & \cdots & n-2 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n-1} \end{array}$$

Note that $\hat{\sigma} \in S_{n-2}$.

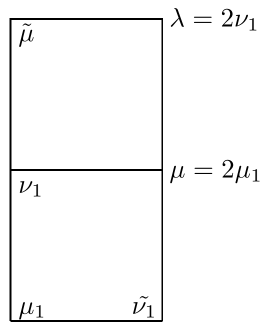
As proved in the proof of Theorem 1, the insertion tableau for $\pi$ can be determined by the shadow lines of the square diagram. Since there is no X in the upper right corner, the X in the uppermost row is paired with the X in row $n-1$ of the $n$th column, thus the first column of the insertion tableau $P$ is $\begin{array}{c} n-1 \\ n \end{array}$. When $P$ is evacuated, the $n-1$ slides down, leaving an empty box in the first column, second row. Thus the first column of $ev(P)$ has an $n$ in the second row, which is the same as the placement of $n$ in $\hat{P}$. At the second step of the evacuation process, the $n-1$ is removed from $P$, leaving an empty square in row one of the first column. Then $ev(P) = \begin{array}{c} n \\ n-1 \end{array} ev(P_{n-2})$, where $P_{n-2}$ is the insertion tableau $P$ without the first column. Since $\hat{P} = \begin{array}{c} n \\ n-1 \end{array} \hat{P}_{n-2}$, the first column of $ev(P(\pi))$ and $\hat{P}(\pi)$ agree. As shown in the proof of Theorem 1, $P(\pi) = \begin{array}{c} n-1 \\ n \end{array} P(\hat{\sigma})$ where $\hat{\sigma}$ is as given above. Then $P_{n-2} = P(\hat{\sigma})$ so $ev(P(\pi)) = \begin{array}{c} n \\ n-1 \end{array} ev(P(\hat{\sigma}))$. Since $\hat{\sigma} \in S_{n-2}$, we can use our inductive hypothesis to obtain

$$ev(P(\pi)) = \begin{array}{c} n \\ n-1 \end{array} ev(P(\hat{\sigma})) = \begin{array}{c} n \\ n-1 \end{array} \hat{P}_{n-2}(\hat{\sigma}) = \hat{P}(\pi).$$

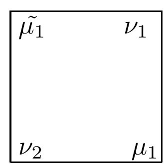
*Case 3:* Suppose the X in column $n$ is in row $a < n-1$. In this case, $\pi$ is given by:

$$\pi = \begin{array}{cccccccc} 1 & 2 & \cdots & i & \cdots & n-1 & n \\ x_1 & x_2 & \cdots & n & \cdots & x_{n-1} & a \end{array}$$

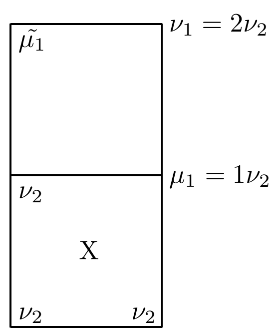
and the upper two squares in the right column of the growth diagram look like:



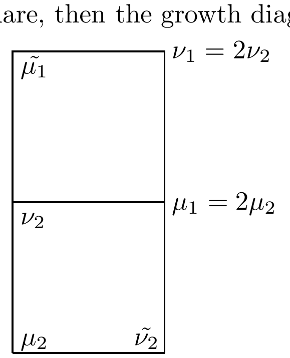

Since $\lambda = 2\nu_1$ and $\mu = 2\mu_1$, then $\lambda$ and $\mu$ differ by the same square as $\nu_1$ and $\mu_1$. If we remove the upper row and rightmost column, as well as any empty rows and columns, then the partial growth diagram of the new upper right square looks like




As before, if there is an X in the new upper right square, then $\nu_1 = 1\mu_1$. If there is an X in the square below this one, then $\nu_1$ and $\mu_1$ differ by a square in the top row of column one:




If there is no X in either square, then the growth diagram looks like



We can continue this procedure until $\mu_i$ and $\nu_i$ differ by a square in the first column, i.e. either $\nu_i = 1\mu_i$ or, if $\mu_i = 1\mu_{i-1}$ for some $\mu_{i-1}$, then $\nu_i = 2\mu_{i-1}$. This implies that $\lambda$ and $\mu$ differ by a square in the $(i+1)$st column. If the square they differ by is in the top row, then $\hat{P}$ has an $n$ in the top row of the $i+1$st column and if they differ by a square in the bottom row of column $i+1$ then $\hat{P}$ has an $n$ in the bottom row of column $i+1$.

We now show that the evacuation tableau $ev(P)$ has an $n$ in the same row of column $i+1$ as $\hat{P}$. If there is not an X in the $n$th row or $(n-1)$st row of the $n$th column of the growth, then by Theorem 1 the first column of $P$ is $\begin{matrix} a \\ n \end{matrix}$, with $a < n-1$.

After removing the $n$th row and the $n$th column and any empty rows and columns from the growth diagram, if there is not an X in one of the top two rows of the rightmost column of the new growth diagram, then the second column of $P$ is $\begin{matrix} b \\ n-1 \end{matrix}$ with $b < n-2$. If, after $i$ iterations of this process, there is an X in the uppermost corner of the growth diagram, then the insertion tableau $P$ has $i$ columns of height 2 followed by a column of height 1. These first $i+1$ columns look like

$$\begin{matrix} a & b & c & \cdots & k & \\ n & n-1 & n-2 & \cdots & n-(i-1) & n-i \end{matrix} \tag{1}$$

with $a < n-1$, $b < n-2$, $\ldots$, $k < n-i$. Thus at the first step of evacuation for $P$, $n-1$ slides one column to the left, $n-2$ slides one column to the left, and so on until $n-i$ slides one column to the left and the evacuation process terminates with an empty box in row 1 of column $i+1$. Thus $ev(P)$ has an $n$ in row 1 of column $i+1$, the same as $\hat{P}$, and after one step of the evacuation procedure the first $i+1$ columns of the $P$ tableau look like:

$$\begin{matrix} a & b & c & \cdots & k & \\ n-1 & n-2 & n-3 & \cdots & n-i & \bullet \end{matrix} .$$

The rest of the $P$ tableau remains unchanged by the evacuation procedure.

After $i$ iterations of this process, if there is an X in the second row from the top, then in the insertion tableau $P$, the first $i+1$ columns have height 2. These first $i+1$ columns look like

$$\begin{matrix} a & b & c & \cdots & k & n-(i+1) \\ n & n-1 & n-2 & \cdots & n-(i-1) & n-i \end{matrix} \tag{2}$$

with $a < n-1$, $b < n-2$, $\ldots$, $k < n-i$. In the evacuation process, $n-1$ through $n-i$ all move one column to the left, $n-(i+1)$ moves one row down and an empty box is left in the second row of column $i+1$. Thus $ev(P)$ has an $n$ in the second row of column $i+1$, as does $\hat{P}$, and after one step of the evacuation procedure the first $i+1$ columns of the $P$ tableau look like:

$$\begin{matrix} a & b & c & \cdots & k & \bullet \\ n-1 & n-2 & n-3 & \cdots & n-i & n-(i+1) \end{matrix} .$$

The part of the $P$ tableau to the right of the $(i+1)$st column remains the same.

Now remove the $n$ in the $(i+1)$st column of $\hat{P}$ to obtain $\hat{P}_{n-1}$ of shape $\mu$. The path tableau $\hat{P}_{n-1}$ is the path tableau obtained from the first $n-1$ rows of the square diagram, which come from the permutation

$$\sigma \quad = \quad \begin{matrix} 1 & 2 & \cdots & i-1 & i & \cdots & n-1 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & a \end{matrix} .$$

Note that $\sigma \in S_{n-1}$. In order to use our inductive hypothesis, it remains to show that after one step of the evacuation of $P$, we obtain $P(\sigma)$. In the proof of Theorem 1, we proved that $P(\pi) = \begin{matrix} a \\ n \end{matrix} \; P(\hat{\sigma})$ where

$$\hat{\sigma} \quad = \quad \begin{matrix} 1 & 2 & \cdots & i-1 & i & \cdots & n-2 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n-1} \end{matrix} .$$

To obtain $P(\sigma)$ we must insert $a$ into $P(\hat{\sigma})$. In (1), $P(\hat{\sigma})$ looks like

$$\begin{matrix} b & c & \cdots & j & k & \\ n-1 & n-2 & \cdots & n-(i-2) & n-(i-1) & n-i \end{matrix}$$

and $a$ inserted into this tableau gives

$$\begin{matrix} a & b & \cdots & j & k \\ n-1 & n-2 & \cdots & n-(i-1) & n-i \end{matrix}$$

for the first $i$ columns and does not change the remaining tableau. This is exactly what $P$ looks like after one step of the evacuation procedure. In (2), $P(\hat{\sigma})$ looks like

$$\begin{matrix} b & c & \cdots & k & n-(i+1) \\ n-1 & n-2 & \cdots & n-(i-1) & n-i \end{matrix}$$

and $a$ inserted into this tableau gives

$$\begin{matrix} a & b & c & \cdots & k & \\ n-1 & n-2 & n-3 & \cdots & n-i & n-(i+1) \end{matrix}$$

for the first $i+1$ columns and does not change the remaining tableau. This is again exactly what $P$ looks like after one step of the evacuation procedure. By induction, $ev(P(\sigma)) = \hat{P}(\sigma)$ and since $P(\pi)$ and $\hat{P}(\pi)$ agree in the $n$th position as well, then $ev(P(\pi)) = \hat{P}(\pi)$.

□

**Corollary 1.**

$$ev(P(\pi^{-1})) = Q(\pi) \qquad ev(P(\pi)) = Q(\pi^{-1})$$

*Proof.* The square diagram for the permutation $\pi^{-1}$ is simply the reflection of the square diagram for the permutation $\pi$ about the line $y = x$ and thus $\hat{P}(\pi) = \hat{Q}(\pi^{-1})$ and $\hat{Q}(\pi) = \hat{P}(\pi^{-1})$. By Theorem 2 we have $\hat{P}(\pi) = ev(P(\pi))$ and $\hat{Q}(\pi) = Q(\pi)$ and the result follows. □

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